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Families of K3 Surfaces in the Smooth Fano 3-folds

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1 Introduction

Varieties assume irreducible, reduced, algebraic and complex.

X : *smooth Fano 3-fold, i.e.*, $-K_X$ is ample.

$B_2(X)$: the second Betti number of X,

$\boldsymbol{P}(\Delta)$: (projective) toric variety associated to a fan $\Delta \subset \boldsymbol{R}^3$.

Classification of smooth Fano 3-folds

$B_2(X) \geq 2$: Mori-Mukai [3][4],

$X = \boldsymbol{P}(\Delta)$: Batyrev [1], Watanabe-Watanabe [5].

Family of $K3$ surfaces

$S \in |-K_X|$: general member. $\rightsquigarrow$ $S : K3$ surface,

$|-K_X|$: family of $K3$ surfaces.

Definition 1 The *Picard lattice* $\mathrm{Pic}(\mathcal{F})$ *of a family* $\mathcal{F}$ *of $K3$ surfaces* is the Picard group of generic member with a cup product.

Definition 2 A surface S is *Gorenstein $K3$* if $K_S \sim 0, h^1(S, \mathcal{O}_S) = 0$ and S has at worst *RDP* (S is birational to a $K3$ surface).

$\mathcal{F}, \mathcal{F}'$: families of $K3$ surfaces in the smooth Fano 3-folds.

PROBLEM. Are the families $\mathcal{F}$ and $\mathcal{F}'$ "generically birationally corresponding" if $\mathrm{Pic}(\mathcal{F}) \simeq \mathrm{Pic}(\mathcal{F}')$?

2 Preliminaries

Fix $\boldsymbol{P}^3 \supset \underset{\text{plane}}{H} \begin{smallmatrix} \supset l & \text{line} \\ \supset C & \text{irreducible smooth cubic} \end{smallmatrix}$

$X' := \mathrm{Bl}_l(\boldsymbol{P}^3)$, $X := \mathrm{Bl}_C(\boldsymbol{P}^3)$.

$\rightsquigarrow$
- X' is toric and X is not toric.
- $\mathrm{Pic}(|-K_{X'}|) \simeq \mathrm{Pic}(|-K_X|)$.
- X depends on the blow-up centre.

PROBLEM. (special case) Are any Gorenstein $K3$ surfaces in $|-K_{X'}|$ birational to Gorenstein $K3$ in $|-K_X|$? (see **Theorem**)

3 Main Result

Families of $K3$ surfaces in the smooth toric Fano 3-folds

We obtain a partial answer to the Problem:

Proposition The Picard lattices of families of $K3$ surfaces in the smooth toric Fano 3-folds are mutually distinct .

$\rightsquigarrow$ Families of $K3$ surfaces in the smooth toric Fano 3-folds are mutually distinct.

Families of $K3$ surfaces $|-K_{X'}|$ and $|-K_X|$

$$\mathcal{F}_1 := \left\{ S_1 \in |-K_{\mathrm{Bl}_l(\boldsymbol{P}^3)}| \, ; S_1 \text{ is Gorenstein } K3 \right\},$$

$$\mathcal{F}_2 := \left\{ S_2 \in \bigcup_{C \subset \boldsymbol{P}^3} |-K_{\mathrm{Bl}_C(\boldsymbol{P}^3)}| \, ; S_2 \text{ is Gorenstein } K3 \right\},$$

where $C \subset \boldsymbol{P}^3$ in the union run all the elliptic curves.

$$V := \left\{ \begin{array}{c} (S, H) \\ \in |-K_{\boldsymbol{P}^3}| \times |\mathcal{O}_{\boldsymbol{P}^3}(1)| \end{array} \middle| \begin{array}{l} H \subset \boldsymbol{P}^3 \ : \text{plane}, \\ S \ : \text{Gorenstein } K3, \\ S \cap H = (\text{line}) \cup (\text{smooth cubic}) \end{array} \right\}.$$

Theorem There exists a correspondence $(V, \mathcal{F}_1, \mathcal{F}_2)$ between the families $\mathcal{F}_1$ and $\mathcal{F}_2$ of $K3$ surfaces. The projections $V \to \mathcal{F}_i$ are surjective. The period domains of the families are isomorphic.

Sketch of proof. $C' \not\simeq C$: elliptic curve.

Step 1. Any Gorenstein $K3$ surface $S_1 \in |-K_{\mathrm{Bl}_C(\boldsymbol{P}^3)}|$ is birational to a Gorenstein $K3$ $S_2 \in |-K_{\mathrm{Bl}_{C'}(\boldsymbol{P}^3)}|$.

Step 2. Any Gorenstein $K3$ surface $S' \in |-K_{X'}|$ is birational to a Gorenstein $K3$ surface $S \in |-K_X|$.

Step 3. Define R by $S_1 \sim_R S_2$ if $S_1 \in \mathcal{F}_1$ is birational to $S_2 \in \mathcal{F}_2$. The relation R defines a correspondence $(V, \mathcal{F}_1, \mathcal{F}_2)$.

4 Remarks

(1) By the Theorem, we are expecting to compute the Gromov-Witten invariants of the family $|-K_X|$ of $K3$ surfaces via the family $|-K_{X'}|$.

(2) We expect another proof of the Theorem via a "small toric degeneration" of non-toric Fano 3-fold X.

(3) We may try to construct correspondences among the 88 families of $K3$ surfaces in the smooth Fano 3-folds.

5 References

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